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On the poisedness of Bojanov–Xu interpolation

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Abstract

In this paper, we consider the bivariate Hermite interpolation introduced by Bojanov and Xu [SIAM J. Numer. Anal. 39(5) (2002) 1780–1793]. The nodes of the interpolation with $\Pi_{2k-\delta}$, where $\delta = 0$ or 1, are the intersection points of $2k+1$ distinct rays from the origin with a multiset of $k+1-\delta$ concentric circles. Parameters are the values and successive radial derivatives, whenever the corresponding circle is multiple. The poisedness of this interpolation was proved only for the set of equidistant rays [Bojanov and Xu, 2002] and its counterparts with other conic sections [Hakopian and Ismail, East J. Approx. 9 (2003) 251–267]. We show that the poisedness of this $(k+1-\delta)(2k+1)$ dimensional Hermite interpolation problem is equivalent to the poisedness of certain $2k+1$ dimensional Lagrange interpolation problems. Then the poisedness of Bojanov–Xu interpolation for a wide family of sets of rays satisfying some simple conditions is established. Our results hold also with above circles replaced by ellipses, hyperbolas, and pairs of parallel lines.

Next a conjecture [Hakopian and Ismail, J. Approx. Theory 116 (2002) 76–99] concerning a poisedness relation between the Bojanov–Xu interpolation, with set of rays symmetric about x-axis, and certain univariate lacunary interpolations is established. At the end the poisedness for a wide class of lacunary interpolations is obtained.

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1. Introduction

Let us denote by π_n the space of univariate polynomials of degree $\leqslant n$. The spaces of bivariate polynomials of total degree $\leqslant n$ and homogeneous polynomials of degree $= n$ are denoted by Π_n and Π_n°, respectively. The unit circumference is

$$S^1 := \{(x, y) \in R^2 : x^2 + y^2 = 1\}.$$

Instead of the radial derivative:

$$D_r f(x, y) := \frac{x}{\sqrt{x^2 + y^2}} f_x(x, y) + \frac{y}{\sqrt{x^2 + y^2}} f_y(x, y)$$

one can use here equivalently a modified one (see [5]):

$$\tilde{D}_r f(x, y) := x f_x(x, y) + y f_y(x, y).$$

We have

$$\tilde{D}_r p = np \quad \text{if} \quad p \in \Pi_n^\circ. \tag{1}$$

The bivariate interpolation discussed here was introduced by Bojanov and Xu [1]. The nodes of this interpolation are the points of intersection of $2k + 1$ distinct rays from the origin with a (multi)set of concentric circles, centered at the origin. Thus they are identified if we have the intersection points of the above rays with the unit circle S^1, called *basic nodes*, and radii of the circles. The set of the basic nodes is denoted by

$$\mathcal{B} := \mathcal{B}_{2k} = \{(x_i, y_i)\}_{i=0}^{2k} \subset S^1.$$

As we will see later the poisedness of the interpolation depends solely on the basic nodes.

The Bojanov–Xu interpolations slightly differ depending on whether the degree of the polynomial class is even or odd. The pair of even and odd degree interpolations with Π_{2k-1} and Π_{2k} is studied simultaneously. For this we introduce a quantity δ and say that the polynomial space is $\Pi_{2k-\delta}$, where δ is 0 or 1, in the above even or odd cases, respectively. Similarly we express statements concerning the above two interpolations in a unique formulation by using δ. In both cases the number of basic nodes is the same: $2k + 1$, while the number of concentric circles, counting the multiplicities, is $k + 1 - \delta$.

Suppose the multiset of the concentric circles consists of s distinct circles with radii $0 < r_1 < \cdots < r_s$ and corresponding multiplicities $\mu_1, \ldots, \mu_s$. Then we have

$$\sum_{i=1}^{s} \mu_i = k + 1 - \delta. \tag{2}$$

The multiplicity of a node equals to the multiplicity of the circle to which it belongs. The interpolation parameters are the values of a function and its radial derivatives up to the order $\mu - 1$, where μ is the multiplicity of the node.

Now we are in a position to formulate for $\delta = 0$ or 1:

Problem 1(δ). Suppose that a set of basic nodes $\mathcal{B}$, a sequence of radii $0 < r_1 < \cdots < r_s$, and multiplicities $\mu_1, \ldots, \mu_s$, that satisfy (2) are given. Then for any given data $\{c_{ijl}\}$ find a (unique) polynomial $p \in \Pi_{2k-\delta}$ such that

$$(\tilde{D}_r)^l p(r_i x_j, r_i y_j) = c_{ijl}, \tag{3}$$

where $1 \leqslant i \leqslant s,\ 0 \leqslant j \leqslant 2k,$ and $0 \leqslant l \leqslant \mu_i - 1.$

The next theorem of Bojanov and Xu [1] provides the poisedness of this interpolation problem in the case of equidistant basic nodes.

Theorem 2 (*Bojanov and Xu [1]*). *Suppose the basic nodes are equidistant on* S^1, $\delta = 0$ *or* 1. *Suppose also that an arbitrary sequence of radii* $0 < r_1 < \cdots < r_s$, *and multiplicities* $\mu_1, \ldots, \mu_s$, *that satisfy* (2) *are given. Then for any given data* $\{c_{ijl}\}$ *there exists a unique polynomial* $p \in \Pi_{2k-\delta}$ *satisfying the condition* (3).

It should be noted that the poisedness of this interpolation cannot be established as readily as in the most cases of known bivariate (multivariate) interpolations. The reason is that in this case there are not enough many points on algebraic curves, particularly on straight lines, or conic sections, to imply the Bézout factorization.

Let us mention that the poisedness of the above interpolation, in the case of one multiple circle, i.e., $s = 1$, was proved later independently in [3], by a factorization method. There a connection was found between the poisedness of the above interpolation, with nodes symmetric about x-axis, and certain univariate lacunary interpolations (see Conjecture 14, Section 3). In [2] the factorization method, which allows to combine the Bojanov–Xu interpolation with other poised interpolations, was extended to the general equidistant case. As it is pointed out there, Theorem 2 holds as well in the case of concentric ellipses. In [4] this result was extended also to other conic sections: concentric or cofocused ellipses, hyperbolas and a single multiple parabola.

In this paper, we will establish the poisedness of Bojanov–Xu interpolation for a wide family of sets of basic nodes $\mathcal{B}$ lying on conic sections (ellipses, hyperbolas, and pairs of parallel lines) centered at the origin.

2. Preliminaries

We study the Bojanov–Xu interpolation with the basic nodes lying on conic section centered at the origin, that is,

$$\mathcal{B} := \mathcal{B}_{2k} \subset C_2^\circ,$$

where

$$C_2^\circ := \{(x, y) : \alpha x^2 + \beta xy + \gamma y^2 = 1\}. \tag{4}$$

These conic sections, after a suitable rotation of axes, are reduced to the following ellipses, hyperbolas, and the pairs of lines given by the equations

$$E_2: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{5}$$

$$H_2: \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{6}$$

$$L_2: \quad y^2 = b^2 \quad \text{or} \quad y = \pm b, \tag{7}$$

respectively, where $a, b > 0$.

We can represent the set of the interpolation nodes as a union of sets which are scaled $\mathcal{B}$ sets, i.e.,

$$\mathcal{N} := \cup_{i=1}^{s} \{r_i \mathcal{B}\}. \tag{8}$$

We call r_i scale constants or "radii". As earlier, we attach multiplicity μ_i to the nodes from $\{r_i \mathcal{B}\}$, $i = 1, \ldots, s$, so that the condition (2) is satisfied. Now the formulation of Bojanov–Xu interpolation in this case proceeds as in Problem 1(δ).

Let us now start the discussion by considering some properties of the Lagrange interpolation with bivariate homogeneous polynomials, which will be needed in the sequel.

Suppose the node set

$$\mathcal{B}_n = \{(x_i, y_i)\}_{i=0}^{n} \subset R^2$$

does not contain collinear nodes (vectors). Then for any function f defined on $\mathcal{B}_n$ there is a unique homogeneous polynomial $P_{n,f}^{\circ} \in \Pi_n^{\circ}$, such that

$$P_{n,f}^{\circ}(x_i, y_i) = f(x_i, y_i), \quad i = 0, \ldots, n. \tag{9}$$

We have the following Lagrange formula

$$P_{n,f}^{\circ}(x, y) = \sum_{i=0}^{n} f(x_i, y_i) L_{ni}^{\circ}(x, y), \tag{10}$$

where L_{ni}° are the homogeneous fundamental polynomials. They are given by the formula

$$L_{ni}^{\circ}(x, y) = \frac{\prod_{j \in *}(x_j y - y_j x)}{\prod_{j \in *}(x_j y_i - y_j x_i)}, \tag{11}$$

where the products are over the set

$$* = \{0, \ldots, i-1, i+1, \ldots, n\}.$$

Denote the remainder of this interpolation by

$$R_{n,f}^{\circ} := f - P_{n,f}^{\circ}.$$

Of course then we have

$$R^{\circ}_{n,f} = 0 \quad \text{if} \quad f \in \Pi^{\circ}_n. \tag{12}$$

Now suppose that the interpolation nodes lie on a conic section centered at the origin:

$$\mathcal{B}_n \subset C^{\circ}_2, \tag{13}$$

where C°_2 is given in (4). Then interestingly, in addition to (12), we have

$$R^{\circ}_{n,f}(\xi, \eta) = 0 \quad \text{if} \quad f \in \Pi^{\circ}_m,\ n - m \geqslant 0 \ \text{ is even and } \quad (\xi, \eta) \in C^{\circ}_2. \tag{14}$$

This, on account of (12), follows readily from the following

Proposition 3. *The remainder $R^{\circ}_{n,f}(\xi, \eta)$, for any $f \in \Pi^{\circ}_m$, and $(\xi, \eta) \in C^{\circ}_2$, is a linear combination of $R^{\circ}_{n,x^i y^j}(\xi, \eta)$ with $i + j = m + 2l,\ l > 0$. Or, more precisely,*

$$R^{\circ}_{n,f}(\xi, \eta) = \sum_{i=0}^{m+2l} c_i R^{\circ}_{n,x^i y^{m+2l-i}}(\xi, \eta) \quad \textit{if} \quad f \in \Pi^{\circ}_m,\ l > 0,\ \textit{and} \ (\xi, \eta) \in C^{\circ}_2, \tag{15}$$

where the coefficients c_i depend only on f, l, and C°_2 (not on the nodes, (ξ, η), or n).

Indeed, suppose $f \in \Pi^{\circ}_m$. Consider the polynomial

$$F(x, y) := (\alpha x^2 + \beta xy + \gamma y^2)^l f(x, y) \in \Pi^{\circ}_{m+2l}.$$

According to (4) we have

$$f(\xi, \eta) = F(\xi, \eta) \quad \text{if} \quad (\xi, \eta) \in C^{\circ}_2.$$

In view of (13) this forces

$$P^{\circ}_{n,f}(x, y) \equiv P^{\circ}_{n,F}(x, y).$$

Now, by using the last two relations we get finally

$$R^{\circ}_{n,f}(\xi, \eta) = R^{\circ}_{n,F}(\xi, \eta) = \sum_{i=0}^{m+2l} c_i R^{\circ}_{n,x^i y^{m+2l-i}}(\xi, \eta),$$

where the coefficients c_i are obtained from the following expansion

$$F(x, y) = \sum_{i=0}^{m+2l} c_i x^i y^{m+2l-i}.$$

Consider now the following $(n + 1) \times (m + 1)$ matrix:

$$V^{\circ}_{n,m} = \begin{bmatrix} x_0^m & x_0^{m-1} y_0 & \cdots & y_0^m \\ \cdots & \cdots & \cdots & \\ x_n^m & x_n^{m-1} y_n & \cdots & y_n^m \end{bmatrix}. \tag{16}$$

In the case of $m = n$ this is the Vandermonde matrix of the above homogeneous interpolation:

$$V_n^\circ := V_{n,n}^\circ. \tag{17}$$

Let us mention that

$$\det V_n^\circ \neq 0 \quad \Leftrightarrow \quad \mathcal{B}_n \text{ does not contain collinear nodes.} \tag{18}$$

Denote by $\mathbf{V}(0)$ and $\mathbf{V}(1)$ the generalized Vandermonde matrices of the Bojanov–Xu interpolation, corresponding to the cases $\delta = 0$ and $\delta = 1$, respectively:

$$\mathbf{V}(\delta) := \mathbf{V}_{\mathcal{B},r_1,\mu_1,\ldots,r_s,\mu_s}(\delta).$$

It consists of the following rows:

$$\left(\tilde{D}_r\right)^l \mathcal{R}(r_i x_j, r_i y_j), \qquad 1 \leqslant i \leqslant s,\ 0 \leqslant j \leqslant 2k,\ 0 \leqslant l \leqslant \mu_i - 1,$$

where the row $\mathcal{R}$ is given by

$$\mathcal{R}(x, y) := \left[x, y, \ldots, x^{2k-\delta}, \ldots, y^{2k-\delta}\right].$$

As it follows from the Bojanov–Xu theorem

$$\det \mathbf{V}(\delta) \not\equiv 0 \quad \text{for any fixed} \quad r_1, \ldots, r_s, \tag{19}$$

where $\delta = 0$ or 1. This statement will be used in the proof of the forthcoming basic Theorem 4.

It is convenient to represent $\mathbf{V}(\delta)$ by its partition of homogeneous blocks $V_{2k,n}^\circ$, given in (16). In the Lagrange case, where the node set $\mathcal{N}$ does not contain multiple scaled $\mathcal{B}$, and therefore the sequence of "radii" is $0 < r_1 < \cdots < r_{k+1-\delta}$, we have

$$\mathbf{V}(\delta) = \begin{bmatrix} V_{2k,0}^\circ & r_1 V_{2k,1}^\circ & \cdots & r_1^{2k-\delta} V_{2k,2k-\delta}^\circ \\ \cdots & \cdots & \cdots & \cdots \\ V_{2k,0}^\circ & r_{k+1-\delta} V_{2k,1}^\circ & \cdots & r_{k+1-\delta}^{2k-\delta} V_{2k,2k-\delta}^\circ \end{bmatrix}. \tag{20}$$

Now consider the Hermite case, that is, when some scaled basic node set: $r\mathcal{B}$ has a multiplicity $\mu > 1$. Then, on account of (1), the corresponding rows of $\mathbf{V}(\delta)$ are partitioned into $V_{2k,n}^\circ$ as follows (cf. [6]):

$$\begin{bmatrix} V_{2k,0}^\circ & r V_{2k,1}^\circ & r^2 V_{2k,2}^\circ & \cdots & r^{2k-\delta} V_{2k,2k-\delta}^\circ \\ 0 & r V_{2k,1}^\circ & 2 r^2 V_{2k,2}^\circ & \cdots & (2k-\delta) r^{2k-\delta} V_{2k,2k-\delta}^\circ \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & r V_{2k,1}^\circ & 2^{\mu-1} r^2 V_{2k,2}^\circ & \cdots & (2k-\delta)^{\mu-1} r^{2k-\delta} V_{2k,2k-\delta}^\circ \end{bmatrix}. \tag{21}$$

The following matrix of order $m + 1$, $0 \leqslant m \leqslant k - 1$, of remainder entries, plays an important role in this paper:

$$A_m = \begin{bmatrix} R_{n,g_0}^\circ(x_{n+1}, y_{n+1}) & \cdots & R_{n,g_m}^\circ(x_{n+1}, y_{n+1}) \\ \cdots & \cdots & \cdots \\ R_{n,g_0}^\circ(x_{2k}, y_{2k}) & \cdots & R_{n,g_m}^\circ(x_{2k}, y_{2k}) \end{bmatrix}, \tag{22}$$

where $n = 2k - m - 1$ and $g_i = x^{m-i} y^i$.

3. Results

We start with the result concerning the factorization of generalized Vandermonde determinant of Bojanov–Xu interpolation with basic nodes lying on a conic section centered at the origin, C_2°, given in (4).

Theorem 4. *Suppose that the basic nodes lie on a conic section centered at the origin*: $\mathcal{B} \subset C_2^\circ$, $\delta = 0$ *or* 1. *Suppose also that a sequence of scale constants* $0 < r_1 < \cdots < r_s$, *and multiplicities* $\mu_1, \ldots, \mu_s$, *that satisfy* (2) *are given. Then* $\det \mathbf{V}(\delta) = 0$ *if* $\mathcal{B}$ *contains* $\delta + 1$ *distinct pairs of opposite nodes. Otherwise, assuming that* $\mathcal{B}_{2k-1}$ *does not contain opposite nodes, we have*

$$\det \mathbf{V}(\delta) = \rho_\delta \prod_{i=k}^{2k-\delta} \det(V_i^\circ) \prod_{i=0}^{k-1} \det(A_i), \tag{23}$$

where the constant $\rho_\delta = \rho_\delta(r_1, \mu_1, \ldots, r_s, \mu_s) \neq 0$.

Let us mention that the statement on $\det \mathbf{V}(\delta) = 0$ immediately follows from Bézout's factorization. Indeed, suppose $\mathcal{B}$ contains $\delta + 1$ distinct pairs of opposite nodes. Then it is easily seen that in the case $\delta = 0$ there is a line passing through $2k + 2$ interpolation nodes (parameters) while in the case $\delta = 1$ there are two lines each passing through $2k$ interpolation nodes.

Next, notice that likewise the Bézout factorization implies that if there is an opposite pair of nodes, say $(x_{2k-1}, y_{2k-1}) = -(x_{2k}, y_{2k})$, then Problem 1(1) is poised with Π_{2k-1} and $\mathcal{B}_{2k}$ if and only if Problem 1(0) is poised with Π_{2k-2} and $\mathcal{B}_{2k-2}$.

Indeed, suppose $\delta = 1$ and a polynomial $p \in \Pi_{2k-1}$ satisfies the homogeneous conditions (3), that is, the conditions (3), with $c_{ijl} = 0$. Then we get by Bézout's theorem that

$$p(x, y) = (ax + by)q(x, y),$$

where the line $ax + by = 0$ passes through the above opposite nodes. From this we conclude that the polynomial $q \in \Pi_{2k-2}$ satisfies the homogeneous conditions (3) corresponding to the set of the basic nodes $\mathcal{B}_{2k-2}$ and $\delta = 0$.

Thus from now on when studying the poisedness we can assume, without loss of generality, that $\mathcal{B}$ does not contain opposite nodes. However, sometimes it will be more convenient to consider the general case.

We readily get from Theorem 4 the following relation between the generalized Vandermonde determinants of Problem 1(δ), corresponding to $\delta = 0$ and 1. Let us mention that these problems have the same set of basic nodes, but the sequences of "radii" (and therefore multiplicities) may differ.

Corollary 5. *Suppose that the basic nodes lie on a conic section centered at the origin*: $\mathcal{B} \subset C_2^\circ$. *Suppose also that sequences of scale constants* $0 < r_1(\delta) < \cdots < r_{s(\delta)}(\delta)$, *and multiplicities* $\mu_1(\delta), \ldots, \mu_{s(\delta)}(\delta)$, *that satisfy* (2) *are given for Problem* 1(δ), *where*

$\delta = 0, 1$. *Then assuming that* $\mathcal{B}_{2k-1}$ *does not contain opposite nodes, we have*

$$\det \mathbf{V}(0) = \frac{\rho_0}{\rho_1} \det(V_{2k}^{\circ}) \ \det \mathbf{V}(1),$$

where $\rho_\delta = \rho_\delta \left(r_1(\delta), \mu_1(\delta), \ldots, r_{s(\delta)}(\delta), \mu_{s(\delta)}(\delta) \right) \neq 0$, *is given in* (23).

In particular, Problem 1(1) *is poised if Problem* 1(0) *is poised. Conversely, Problem* 1(0) *is poised if Problem* 1(1) *is poised and* $\mathcal{B}$ *does not contain opposite nodes.*

Let us mention that Corollary 5, as well as the two statements following Theorem 4, are proved in [6] in two special cases. Namely, in the case of one multiple circle, i.e., $s(0) = s(1) = 1$, and the Lagrange case of concentric circles with a special sequence of radii (the same for both Problem 1(0, 1)).

The polynomial interpolation introduced below will be used in the next theorem. Consider the Lagrange interpolation with the set of basic nodes $\mathcal{B}_{2k}$ and bivariate monomials of two distinct total degrees: m, n, with $m+n = 2k-1$. Hence the pair of degrees always consists of even and odd numbers. We call this interpolation $\{m, n\}$-degree interpolation. Thus the space of the $\{m, n\}$ interpolating polynomials is

$$\Pi_m^{\circ} \oplus \Pi_n^{\circ} = \left\{ p : p(x, y) = \sum_{i=0}^{m} \alpha_i x^i y^{m-i} + \sum_{i=0}^{n} \beta_i x^i y^{n-i} \right\}.$$

The following theorem reduces the poisedness of Bojanov–Xu interpolation, which is of Hermite type, to the poisedness of the above $\{m, n\}$-degree Lagrange interpolations. Let us mention also that the Bojanov–Xu interpolation is over $\Pi_{2k-\delta}$, i.e., a polynomial space of dimension $(k + 1 - \delta)(2k + 1)$, while all the above $\{m, n\}$-degree interpolations are over polynomial spaces of dimension just $2k + 1$. Note also that the set of nodes of all these latter interpolations is $\mathcal{B}$.

Theorem 6. *Suppose that the basic nodes, scale constants, multiplicities and* δ *are as in Theorem* 4. *Suppose also that* $\mathcal{B}$ *does not contain* $\delta + 1$ *pairs of opposite nodes. Then the Bojanov–Xu interpolation with* $\Pi_{2k-\delta}$ *is poised if and only if the* $\{m, n\}$*-degree Lagrange interpolations,* $m = 0, \ldots, k - 1, \ n = 2k - m - 1$, *are poised with the basic node set* $\mathcal{B}$.

Regarding the $\{k - 1, k\}$-degree interpolation above let us mention the following

Proposition 7. *The* $\{k - 1, k\}$*-degree interpolation is always poised for* E_2 *and* H_2, *given in* (5)–(6). *While for* L_2, *given in* (7), *the interpolation is poised if and only if out of* $2k + 1$ *interpolating nodes* $k + 1$ *belong to one line and remaining* k *to another.*

Indeed, for the cases E_2, H_2, or the direct implication for L_2, assume that $p \in \Pi_{k-1}^{\circ} \oplus \Pi_k^{\circ}$ vanishes at the $(2k + 1)$ nodes of $\mathcal{B} \subset C_2^{\circ}$. Then by the Bézout theorem we get

$$p(x, y) = (\alpha x^2 + \beta xy + \gamma y^2 - 1) r(x, y).$$

Now notice that, unless $r \equiv 0$ the difference of maximum and minimum total degrees of monomials of the polynomial p is at least 2, which is a contradiction.

To verify the inverse implication for L_2 suppose that one line, say $y-b=0$, contains $\geqslant k+2$ nodes and the another $\leqslant k-1$ nodes: $(x_i, y_i),\ i=1,\dots,s$, where $s\leqslant k-1$. Then the following nonzero polynomial

$$y^{k-1-s}(y-b)\prod_{i=1}^{s}(x_i y - y_i x) \in \Pi_{k-1}^{\circ} \oplus \Pi_k^{\circ}$$

vanishes at all interpolation nodes.

The following property of the $\{m, n\}$-degree interpolation is interesting in connection with the analog property of the Bojanov–Xu interpolation mentioned after Theorem 4. Below, by the $\{-1, 2k-2\}$-degree interpolation we mean the Lagrange interpolation with Π_{2k-2}°.

Proposition 8. *Let $\mathcal{B}$ contains a collinear pair of nodes, say (x_{2k-1}, y_{2k-1}) and (x_{2k}, y_{2k}). Suppose that there is no other node collinear with this pair. Then the $\{m, n\}$-degree interpolation, $m+n=2k-1$, is poised with $\mathcal{B}_{2k}$ if and only if the $\{m-1, n-1\}$-degree interpolation is poised with $\mathcal{B}_{2k-2}$.*

Indeed, let

$$(g-f)(x_i, y_i) = 0 \quad \text{for} \quad i = 0, \dots, 2k,$$

where $g \in \Pi_m^{\circ}$ and $f \in \Pi_n^{\circ}$. Suppose $(x_{2k-1}, y_{2k-1}) = \lambda(x_{2k}, y_{2k})$. Then we have

$$g(x_{2k}, y_{2k}) = f(x_{2k}, y_{2k}), \quad \text{and} \quad \lambda^m g(x_{2k}, y_{2k}) = \lambda^n f(x_{2k}, y_{2k}).$$

This implies $(\lambda^m - \lambda^n)g(x_{2k}, y_{2k}) = 0$. Since $\lambda \neq 0, 1$, and $m+n = 2k-1$, we get $g(x_{2k}, y_{2k}) = 0$ therefore $f(x_{2k}, y_{2k}) = 0$ too. Consequently

$$f(x, y) = (ax+by)f_1(x, y) \quad \text{and} \quad g(x, y) = (ax+by)g_1(x, y),$$

where $g_1 \in \Pi_{m-1}^{\circ}$ and $f_1 \in \Pi_{n-1}^{\circ}$ and the line with the equation $ax+by=0$ passes through (x_{2k}, y_{2k}). Finally, in view of the second hypothesis on $\mathcal{B}$ we get

$$(g_1 - f_1)(x_i, y_i) = 0 \quad \text{for} \quad i = 0, \dots, 2k-2.$$

It is easily seen from Theorem 6 (or 3):

Corollary 9. *The poisedness of Bojanov–Xu interpolation depends solely on the basic nodes $\mathcal{B}$.*

Thus the poisedness does not depend on “radii”, i.e., scale constants r_i, or multiplicities μ_i of the node set $\mathcal{N}$ given in (8).

In the next theorem we establish the poisedness of a wide class of Bojanov–Xu interpolations satisfying certain simple conditions. Let us start with notation and a definition.

We denote by $\widehat{N, M}$, where $N, M \in C_2^{\circ}$ are not collinear, the arc of C_2° with endpoints N and M. Note that in the case of ellipse (circle), there are two such arcs. Then we choose the one with smaller angle from the origin. By $(\widehat{N, M})$, we denote the open arc, i.e., $N, M \notin (\widehat{N, M})$.

Definition 10. We say that the set of basic nodes

$$\mathcal{B} = \{(x_i, y_i)\}_{i=0}^{2k} \subset C_2^\circ$$

satisfies the *opposite node* property, if there is a subset $\mathcal{B}_k^* = \{N_i\}_{i=0}^{k}$ of $k+1$ nodes from $\mathcal{B}$ such that

(a) The nodes $\{N_i\}_{i=0}^{k}$ are lying successively on a continuous arc Γ of the conic section C_2°,
(b) The arc Γ is on one side of some line passing through the origin,
(c) for each arc $\gamma_i := \widehat{N_{i-1}, N_i}$, there is a node $N_{k+i} \in \mathcal{B} \setminus \mathcal{B}_k^*$ such that

$$-N_{k+i} \in (\gamma_i), \; i = 1, \ldots, k.$$

Let us mention that one can readily construct node set $\mathcal{B} \in C_2^\circ$ possessing the above property. In the case of ellipse E_2, given in (5), choose a line passing through the origin and $k+1$ nodes of $\mathcal{B}_k^*$ on the ellipse lying on one side of the line. Then the remaining k nodes are chosen satisfying the above condition (c). In the case of the hyperbola H_2 or the pair of lines L_2, given in (6)–(7), we choose $k+1$ nodes of $\mathcal{B}_k^*$ on one branch of the hyperbola or on one of lines, respectively. Then the remaining k nodes are chosen on the other branch or line such that the condition (c) is satisfied.

Now we are in a position to formulate

Theorem 11. *Suppose that the basic nodes, scale constants, multiplicities, and δ are as in Theorem* 4. *Suppose also that $\mathcal{B} \subset C_2^\circ$ does not contain $\delta+1$ pairs of opposite nodes and satisfies the opposite node property. Then the Bojanov–Xu interpolation with $\Pi_{2k-\delta}$ is poised*.

The next result concerns the ellipse E_2 given in (5). By the angle between two noncollinear points (vectors) we mean the one that is less than π.

Corollary 12. *Suppose that the set of basic nodes lies on the ellipse*: $\mathcal{B} \subset E_2$ *and does not contain $\delta+1$ pairs of opposite nodes, while the scale constants, multiplicities, and δ are as in Theorem* 4. *Suppose also that all the angles α between any two neighbor basic nodes satisfy the inequality*

(i) $\alpha \leqslant \pi/k$, *or all they satisfy the inequality,*
(ii) $\alpha \geqslant \pi/(k+1)$.

Then the Bojanov–Xu interpolation with $\Pi_{2k-\delta}$ is poised.

Next we turn to the conjecture presented in [3]. Here the $2k+1$ basic nodes lying on E_2 or H_2, given by (5)–(6), are symmetric about x-axis. Therefore one of them lies on x-axis and coincides with $(\pm a, 0)$. For certainty let us take $(a, 0)$. Denote the set of the basic nodes in this case by

$$\mathcal{B}_\pm := \{x_i, \pm y_i)\}_{i=0}^{k},$$

where $(x_0, y_0) = (a, 0)$. The projections of these nodes on x-axis give rise to the following knot set:

$$\mathcal{X} = \{x_i\}_{i=0}^{k},$$

where $x_0 = a$.

Denote

$$I_{m,n} := \{0 \leqslant i \leqslant n : i \neq n-1, n-3, \ldots, m+2\}.$$

Consider the following lacunary interpolation problems for $\varepsilon = 0$ or 1.

Problem 13(ε). Let $0 \leqslant m \leqslant k-1,\ n = 2k-m-1$, and the set of knots $\mathcal{X}$ be given. Then for any given data $\{c_i\}$ find a (unique) polynomial q_ε of form $q_\varepsilon(x) = \sum_{i \in I_{m,n}} \alpha_i x^{i-\varepsilon}$, such that

$$q_\varepsilon(x_i) = c_i \quad \text{for} \quad i = \varepsilon, \ldots, k. \tag{24}$$

It was shown in [3], in the case of one multiple circle, i.e., $s = 1$, that if the above univariate lacunary interpolations are poised with $\mathcal{X}$ then the Bojanov–Xu interpolation is poised with $\mathcal{B}_\pm$. It was conjectured in [3, Remark 13(iii)] that the reverse statement also is true:

Conjecture 14 (*Hakopian and Ismail [3]*). *Suppose that the basic nodes are symmetric about x-axis and lie on the unit circle*: $\mathcal{B}_\pm \subset S^1$. *Then the Bojanov–Xu interpolation with* $\Pi_{2k-\delta},\ \delta = 0, 1$, *and one multiple circle is poised if and only if there are no opposite knots in $\mathcal{X}$ and all the univariate lacunary interpolations in Problem* 13(ε), $\varepsilon = 0, 1$, *are poised*.

We show that the above Conjecture holds in more general case. Namely for arbitrary multiset $\mathcal{N}$ with the set of basic nodes $\mathcal{B}_\pm$ on E_2 or H_2, given in (5)–(6).

Theorem 15. *Suppose that the basic nodes are symmetric about x-axis and lie on the ellipse or hyperbola*: $\mathcal{B}_\pm \subset E_2$ or H_2. *Suppose also that the scale constants, multiplicities, and δ are as in Theorem* 4. *Then the Bojanov–Xu interpolation with $\Pi_{2k-\delta}$ is poised if and only if there are no opposite knots in $\mathcal{X}$ and all the univariate lacunary interpolations in Problem* 13(ε), $\varepsilon = 0, 1$, *are poised*.

Note that if $\mathcal{X}$ contains two opposite knots then $\mathcal{B}_\pm$ contains two pairs of opposite nodes. Consequently, according to Theorem 4, the Bojanov–Xu interpolation with $\Pi_{2k-\delta}$ is not poised, where $\delta = 0$ or 1. Thus, according to Theorem 5, to prove the above theorem it suffices to prove

Lemma 16. *Let* $0 \leqslant m \leqslant k-1$. *Suppose that $\mathcal{X}$ does not contain opposite knots. Then the* $\{m, n\}$*-degree interpolation, where* $m < n,\ n = 2k-m-1$, *is poised with* $\mathcal{B}_\pm \subset E_2$ *or* H_2 *if and only if the two lacunary interpolations in Problem* 11(ε), *corresponding to* $\varepsilon = 0, 1$, *are poised.*

Now let us consider the opposite node property for the case of $\mathcal{B}_\pm$. Note that then the condition $-N_{k+i} \in (\gamma_i)$ in Definition 10(c) is reducing to

$$x_{i-1} < -x_{k+i} < x_i,$$

whenever $x_{i-1} \neq x_i$, where $N_j = (x_j, y_j)$. On account of this we get that the opposite node property is satisfied with $\mathcal{B}_\pm \subset E_2$ provided

$$\begin{aligned} &a = x_0 > -x_k > x_1 > -x_{k-1} > x_2 > \cdots > x_s > -x_{s+1} > 0, \qquad \text{or} \\ &a = x_0 > -x_k > x_1 > -x_{k-1} > x_2 > \cdots > x_{s-1} > -x_{s+1} > x_s > 0, \end{aligned} \tag{25}$$

for $k = 2s+1$ or $k = 2s$, respectively. In the case of hyperbola: $\mathcal{B}_\pm \subset H_2$ the corresponding conditions are

$$\begin{aligned} &a = x_0 < -x_k < x_1 < -x_{k-1} < x_2 < \cdots < x_s < -x_{s+1}, \qquad \textit{or} \\ &a = x_0 < -x_k < x_1 < -x_{k-1} < x_2 < \cdots < x_{s-1} < -x_{s+1} < x_s, \end{aligned} \tag{26}$$

for $k = 2s+1$ or $k = 2s$, respectively.

Indeed, it can be readily verified that the set B_k^* of Definition 10 can be chosen as follows:

$$B_k^* = \{(x_i, \pm y_i)\}_{i=s+1}^k \quad \text{or} \quad \{(x_i, \pm y_i)\}_{i=0}^s,$$

for $k = 2s+1$ or $k = 2s$, respectively, where $y_i := b\sqrt{\pm\left(1 - \frac{(x_i)^2}{a^2}\right)}, \quad i = 0, \ldots, k$. Therefore, taking into account also Theorems 11 and 15, we get the following result on the poisedness of univariate lacunary interpolations.

Corollary 17. *Suppose the chain of inequalities* (25) (*or* (26)) *holds for the knot set* $\mathcal{X}$. *Then all the lacunary interpolations in Problem* 13(ε), $\varepsilon = 0, 1$, *are poised.*

Now let us turn again to the lacunary interpolation Problem 13(ε). Notice that the interpolations there are clearly poised in the case of nonnegative knots: $x_i \geqslant 0,\ i = 0, \ldots, k$. In fact this readily follows from the Descartes signs rule (see [7, Part 5]). Now, going back from these interpolation problems to the Bojanov–Xu interpolation, on account of Theorem 15, we get

Corollary 18. *Suppose that the basic nodes are in the first or fourth quarter, symmetric about x-axis, and lying on the ellipse or hyperbola*: $\mathcal{B}_\pm \subset E_2$ *or* H_2. *Suppose also that the scale constants, multiplicities and δ are as in Theorem* 4. *Then the Bojanov–Xu interpolation with $\Pi_{2k-\delta}$ is poised.*

4. Proofs

Proof of Theorem 4. Proof of Theorem 4 consists of two parts corresponding to the cases $\delta = 1$ or 0. Part 1 is the main one to which Part 2 will be reduced readily.

Part 1: The case $\delta = 1$. The proof of this part consists of three steps: Proposition 19, factorizations V°, and factorizations A. In Step 1 we bring the Vandermonde matrix of

Bojanov–Xu interpolation, by using the elementary row operations, into a special form. Both Steps 2 and 3 contain k factorizations (cf. formula (23)). In Step 2 the factors are the Vandermonde determinants of homogeneous interpolations: $\det(V_i^\circ), \quad i = k, \ldots, 2k-1$ (see (16)–(17)). In Step 3 the factors are $\det(A_i), \quad i = 0, \ldots, k-1$, whose entries are remainders of the homogeneous interpolation (see (22)).

Step 1: Consider the Vandermonde matrix of Bojanov–Xu interpolation corresponding to $\delta = 1 : \mathbf{V}(1)$. In the next proposition it will be transformed to the following form

$$\mathbf{V}' = \begin{bmatrix} r_{10} V_{2k,0}^\circ & & \cdots & & r_{1,2k-2} V_{2k,2k-2}^\circ & V_{2k,2k-1}^\circ \\ r_{20} V_{2k,0}^\circ & \cdots & & r_{2,2k-3} V_{2k,2k-3}^\circ & V_{2k,2k-2}^\circ & 0 \\ \cdots & & & \cdots & & \cdots \\ r_{k0} V_{2k,0}^\circ & \cdots \; r_{k,k-1} V_{2k,k-1}^\circ & V_{2k,k}^\circ & 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

where r_{ij} depends on $r_1, \mu_1, \ldots, r_s, \mu_s$.

Proposition 19. *One can bring the matrix $\mathbf{V}(1)$, by using the elementary row operations, with constants depending on $r_1, \mu_1, \ldots, r_s, \mu_s$, to the above matrix $\mathbf{V}'$. Moreover, the constants used in the operation of multiplication of a row are not zero. Therefore*

$$\det \mathbf{V}(1) = \rho' \det \mathbf{V}', \tag{28}$$

where ρ' depends only on $r_1, \mu_1 \ldots, r_s, \mu_s$.

We will use the partitioned form of $\mathbf{V}(1)$ into homogeneous submatrices (see (20)–(21)). Let us implement the above-mentioned operations first in the Lagrange case, that is, when there are no multiple scaled basic sets. Next it will be modified to fit the general Hermite case of multiple scaled basic sets, too. Thus we start with the Vandermonde determinant $\mathbf{V}(1)$ in the Lagrange form (20). We are going to use the Gauss (block) elimination to reduce it to the form (27). The smoothest way for this is through the use of divided differences. For this purpose we first factor out $(r_l)^{2k-1}$ from the lth block row of the matrix $\mathbf{V}(1)$, for $l = 1, \ldots, k$, and set $t_l := \frac{1}{r_l}$. This reduces $\mathbf{V}(1)$ to

$$\begin{bmatrix} t_1^{2k-1} V_{2k,0}^\circ & t_1^{2k-2} V_{2k,1}^\circ & \cdots & V_{2k,2k-1}^\circ \\ \cdots & \cdots & \cdots & \cdots \\ t_k^{2k-1} V_{2k,0}^\circ & t_k^{2k-2} V_{2k,1}^\circ & \cdots & V_{2k,2k-1}^\circ \end{bmatrix}.$$

Next we replace successively the block rows of the above matrix, starting with the last one, by block rows with coefficients expressed by divided differences. Namely the lth block row above is replaced by the row

$$\begin{bmatrix} t_{l0} V_{2k,0}^\circ & t_{l1} V_{2k,1}^\circ & \cdots & t_{l,2k-1} V_{2k,2k-1}^\circ \end{bmatrix}, \tag{29}$$

where

$$t_{l\nu} = [t_1, \ldots, t_l] t^{2k-\nu-1}.$$

Note that this change actually is a result of elementary row operations. Indeed, consider the Lagrange formula with distinct knots

$$[t_1, \dots, t_l]f = \sum_{i=1}^{l} \frac{f(t_i)}{\prod_{j\in *}(t_i - t_j)}, \tag{30}$$

where the products are over the set $* = \{1, \dots, i-1, i+1, \dots, l\}$. According to this formula the above replacement corresponds to the operation of multiplication of the lth block row by $\frac{1}{\prod_*(t_l - t_j)} \neq 0$ and adding to it a linear combination of the first $l-1$ block rows.

Now we use the well-known property of divided differences:

$$[t_1, \dots, t_l]t^v = \begin{cases} 1 & \text{if } v = l-1, \\ 0 & \text{if } v \leqslant l-2. \end{cases}$$

This implies that $t_{l,2k-l} = 1$ and the coefficients next to it are 0, i.e., the above row (29) is equal to

$$\left[t_{l0} V^\circ_{2k,0} \;\; t_{l1} V^\circ_{2k,1} \;\; \cdots \;\; t_{l,2k-l-l} V^\circ_{2k,2k-l-1} \;\; V^\circ_{2k,2k-l} \;\; 0 \; \dots 0 \right].$$

Thus the matrix $\mathbf{V}(1)$ is reduced to $\mathbf{V}'$.

Next let us turn to the general Hermite case of arbitrary multiple scaled basic sets.

Suppose some scaled basic node set, $r\mathcal{B}$, has a multiplicity μ and consider the corresponding rows of $\mathbf{V}(1)$ partitioned into $V^\circ_{2k,n}$, given in (21). Below we will show that one can transform (21), by elementary row operations, into the following form

$$\begin{bmatrix} t^{2k-1} V^\circ_{2k,0} & & \cdots & & t V^\circ_{2k,2k-2} & V^\circ_{2k,2k-1} \\ (2k-1) t^{2k-2} V^\circ_{2k,0} & & \cdots & 2t V^\circ_{2k,2k-3} & V^\circ_{2k,2k-2} & 0 \\ \cdots & & \cdots & & & \cdots \\ \frac{(2k-1)!}{(2k-\mu)!} t^{2k-\mu} V^\circ_{2k,0} & \cdots & \mu! t V^\circ_{2k,2k-\mu-1} & (\mu-1)! V^\circ_{2k,2k-\mu} \; 0 & 0 & 0 \end{bmatrix}, \tag{31}$$

where the block rows are successive derivatives of the first row with respect to t.

Meanwhile let us verify that this transformation solves the problem. Consider the generalized Vandermonde matrix of the Bojanov–Xu interpolation for $\delta = 1$. Suppose the multiplicity associated with “radii” r_i or $t_i = 1/r_i$, is μ_i i.e.,

$$\{\tau_1, \dots, \tau_k\} := \{\underbrace{t_1, \dots, t_1}_{\mu_1}, \dots, \underbrace{t_s, \dots, t_s}_{\mu_s}\},$$

where $k = \mu_1 + \cdots + \mu_s$. Suppose also that the above-mentioned transform is already performed for the block rows of the matrix corresponding to those t_i for which $\mu_i > 1$. This enables us to order the block rows of the matrix in accordance with the above sequence $\{\tau_1, \dots, \tau_k\}$. In particular, the $(l+1)$th block row coincides with the derivative of lth block row with respect to t, whenever $\tau_l = \tau_{l+1}$. Now, as in the above Lagrange case, we replace successively the block rows of the matrix, starting with the last one, by block rows with coefficients expressed by divided differences, whose knots now may be multiple. Namely the lth row is replaced by (29) where

$$t_{l\nu} = [\tau_1, \ldots, \tau_l]t^{2k-\nu-1}.$$

Then we make use of the generalized Lagrange formula for divided differences:

$$[\underbrace{t_1, \ldots, t_1}_{\mu_1}, \ldots, \underbrace{t_s, \ldots, t_s}_{\mu_s}]f = \sum_{i=1}^{s} \sum_{j=0}^{\mu_i - 1} c_{ij} f^j(t_i),$$

where $c_{i,\mu_i-1} \neq 0$. This formula, in the same way as (30) above, leads to the desired result.

Now, regarding the transformation of the matrix (21) to (31), notice that what we need is to bring, by elementary row operations, the following matrix

$$M_1 := \begin{bmatrix} 1 & r & r^2 & \cdots & r^n \\ 0 & r & 2r^2 & \cdots & nr^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & r & 2^{\mu-1}r^2 & \cdots & n^{\mu-1}r^n \end{bmatrix}$$

to the matrix

$$M_2 := \begin{bmatrix} t^n & & \cdots & & t^2 & t & 1 \\ nt^{n-1} & & \cdots & & 2t & 1 & 0 \\ \cdots & & \cdots & & & \cdots & \\ \frac{n!}{(n-\mu+1)!}t^{n-\mu+1} \cdots & \frac{\mu!}{1!}t & (\mu-1)! & 0 & 0 & 0 \end{bmatrix},$$

where $t = 1/r$. Notice that it is enough to transform M_1 to

$$M_3 := \begin{bmatrix} 1 & r & & \cdots & r^{n-2} & r^{n-1} & r^n \\ n & (n-1)r & & \cdots & 2r^{n-2} & r^{n-1} & 0 \\ \cdots & & & \cdots & & \cdots & \\ \frac{n!}{(n-\mu+1)!} & \frac{(n-1)!}{(n-\mu)!}r \;\cdots & \frac{\mu!}{1!}r^{n-\mu} & (\mu-1)!r^{n-\mu+1} & 0 & \cdots & 0 \end{bmatrix},$$

since getting M_2 from M_3 is immediate. Indeed, for this we are to factor out r^n from the first row, r^{n-1} from the second row and so on. What remains then is to set $t = 1/r$.

Next note that clearly it suffices to do the reverse of what we need. Namely, to transform by elementary row operations the matrix M_3 to M_1. This can be done in view of the fact that the lth row of the matrix M_1 is a linear combination of first l rows of M_3 with coefficients depending only on l, $l = 1, \ldots, \mu$.

It is enough to verify the latter only for the last: μth row of M_1. For this observe that the coefficients of the entries of the last row of M_1 and all the μ rows of M_3 coincide respectively with the values of the monomial $x^{\mu-1}$ and following μ polynomials

$$1,\ (n-x),\ (n-x)(n-x-1), \ldots, (n-x)(n-x-1)\cdots(n-x-\mu+2), \tag{32}$$

at the points $0, 1, \ldots, n$. Finally, the only point remains is to represent the monomial $x^{\mu-1}$ as a linear combination of polynomials in (32), which can be readily checked.

Step 2: Factorizations V°. The case when $\mathcal{B}$ contains two or more distinct pairs of opposite nodes was discussed just after the formulation of the theorem. By the way this could be done readily also by using the matrix $\mathbf{V}'$ (see also the corresponding matrix in Part 2). It remains to consider the case when the basic node set contains no more than one such pair of nodes. Thus suppose without loss of generality that

$$\text{there are no collinear nodes in } \mathcal{B}_{2k-1}. \tag{33}$$

Now, we turn to the matrix $\mathbf{V}'$ given in (27). Notice that in the last block column, where there are $2k$ columns, the nonzero entries form a submatrix $V^{\circ}_{2k,2k-1}$ of dimension $(2k+1) \times 2k$.

We are going to implement a basic step — eliminate the last row of $V^{\circ}_{2k,2k-1}$ corresponding to the node (x_{2k}, y_{2k}). After this the above-mentioned submatrix with nonzero entries in the last block column of $\mathbf{V}'$ becomes $V^{\circ}_{2k-1,2k-1}$ which has dimension $2k \times 2k$. This will enable us to factorize $\det \mathbf{V}'$ by using the Laplace expansion along the last block column of the determinant.

Throughout the proof, $\mathcal{R}_i^j$ stands for the jth row inside the ith block row of the matrix under the discussion, where the latter will be clear from the context.

In the above-mentioned elimination we will use homogeneous Lagrange interpolation with the polynomial space Π°_{2k-1} and the nodes $(x_0, y_0), \ldots, (x_{2k-1}, y_{2k-1})$. This interpolation is poised in view of (18) and (33). Let us now perform the elimination by the following row operation inside the first block row of the matrix $\mathbf{V}'$:

$$\mathcal{R}_1^{2k+1} \to \mathcal{R}_1^{2k+1} - \sum_{i=1}^{2k} L^{\circ}_{2k-1,i-1}(x_{2k}, y_{2k}) \mathcal{R}_1^i,$$

where the coefficients are the fundamental polynomials given in (11).

By virtue of the Lagrange formula (10) we get that the old row $\mathcal{R}_1^{2k+1}$:

$$\left[r_{10} \;\; r_{11}x \;\; r_{11}y \ldots r_{1,2k-2}x^{2k-2} \ldots r_{1,2k-2}y^{2k-2} \;\; x^{2k-1} \ldots y^{2k-1} \right] |_{(x_{2k}, y_{2k})}$$

will be replaced by the following new one:

$$\begin{aligned} \mathcal{R}_1^{2k+1} = \big[& r_{10} R^n_{00} \; r_{11} R^n_{10} \; r_{11} R^n_{01} \ldots r_{1,2k-2} R^n_{2k-2,0} \\ & \ldots r_{1,2k-2} R^n_{0,2k-2} \; 0 \ldots 0 \big] |_{(x_{2k}, y_{2k})}, \end{aligned}$$

where $n = 2k - 1$ and

$$R^n_{ij} := R^{\circ}_{n,\phi_{ij}} := \phi_{ij} - P^{\circ}_{n,\phi_{ij}} \text{ and } \phi_{ij} := x^i y^j. \tag{34}$$

Notice that the entries of the above row corresponding to R^n_{ij} with $i + j = 2k - 1$ were eliminated since the monomials ϕ_{ij} there belong to the space of interpolating polynomials: Π°_{2k-1}.

Now one could already factorize the Vandermonde determinant. But in order not to be occupied with the above row $\mathcal{R}_1^{2k+1}$ in the next V°-factorizations we need to eliminate also its entries corresponding to R^n_{ij} with $i + j = 2k - 2, 2k - 3, \ldots, 2k - k = k$. To do this for the case $i + j = 2k - 2$, consider the rows in the second block row of the matrix $\mathbf{V}'$. Let us start by taking the same linear combination here, as in the first block row, and designate it by $\mathcal{R}_2^*$, i.e.,

$$\mathcal{R}_2^* := \mathcal{R}_2^{2k+1} - \sum_{i=1}^{2k} L^\circ_{2k-1,i-1}(x_{2k}, y_{2k})\mathcal{R}_2^i.$$

Notice that this results in

$$\mathcal{R}_2^* = \big[r_{20} R^n_{00} \dots r_{2,2k-3} R^n_{2k-3,0} \dots r_{2,2k-3} R^n_{0,2k-3} \\ \times R^n_{2k-2,0} \dots R^n_{0,2k-2} 0 \dots 0\big] |_{(x_{2k}, y_{2k})}.$$

Then the row operation

$$\mathcal{R}_1^{2k+1} \to \mathcal{R}_1^{2k+1} - r_{1,2k-2}\mathcal{R}_2^*$$

provides the desired elimination. By continuing eliminations this way till the kth block row we will finally reduce $\mathcal{R}_1^{2k+1}$ to the following row, where $n = 2k - 1$ and a_i^1 are some numbers:

$$\mathcal{R}_1^{2k+1} = \left[a_0^1 R^n_{00}\ a_1^1 R^n_{10}\ a_1^1 R^n_{01} \dots a^1_{k-1} R^n_{k-1,0} \dots a^1_{k-1} R^n_{0,k-1}\ 0 \dots 0\right] |_{(x_{2k}, y_{2k})}.$$

Note that the coefficients a_i^1 are the same for the entries R^n_{ij} with $i + j = s$. Let us mention that the entries of these rows, preceding the last zeros, will remain unchanged till the end of this step of factorizations.

Now we get by the Laplace theorem:

$$\det \mathbf{V}' = \rho'' \det \mathbf{V}'' \det V^\circ_{2k-1,2k-1}, \tag{35}$$

where $\rho'' = \rho''(r_1, \mu_1, \dots, r_s, \mu_s)$ and the matrix $\mathbf{V}''$ is obtained from $\mathbf{V}'$ by replacing the first block row by the above row $\mathcal{R}_1^{2k+1}$ and by canceling the last block column ($2k$ columns). In order not to change the numbers of block rows it is convenient to consider the latter row as the first block row of $\mathbf{V}''$ (which has just one row).

Let us then turn to the matrix $\mathbf{V}''$. Notice that in the last block column, where there are $2k - 1$ columns, the nonzero elements form the submatrix $V^\circ_{2k,2k-2}$ of dimension $(2k + 1) \times (2k - 1)$.

Next we implement the analog of above-mentioned basic step — eliminate the last two rows of $V^\circ_{2k,2k-2}$ which correspond to the nodes (x_{2k-1}, y_{2k-1}) and (x_{2k}, y_{2k}). After this step the above-mentioned submatrix with nonzero entries in the last block column of $\mathbf{V}''$ becomes $V^\circ_{2k-2,2k-2}$ which has dimension $(2k - 1) \times (2k - 1)$. This will enable us to use the Laplace theorem for another factorization.

For the elimination, as earlier, we will use homogeneous Lagrange interpolation. Here the polynomial space is Π°_{2k-2} and the nodes are $(x_0, y_0), \dots, (x_{2k-2}, y_{2k-2})$. This interpolation is poised in view of (18) and (33). Now let us do the following row operations inside the second block row

$$\mathcal{R}_2^{2k+1} \to \mathcal{R}_2^{2k+1} - \sum_{i=1}^{2k-2} L^\circ_{2k-2,i-1}(x_{2k}, y_{2k})\mathcal{R}_2^i,$$

$$\mathcal{R}_2^{2k} \to \mathcal{R}_2^{2k} - \sum_{i=1}^{2k-2} L^\circ_{2k-2,i-1}(x_{2k-1}, y_{2k-1})\mathcal{R}_2^i.$$

By using the Lagrange formula (10) we get that the old $2k$th and $(2k+1)$th rows of the second block row will be replaced by the following new ones:

$$\mathcal{R}_2^{2k+1} = \big[r_{20} R_{00}^n \;\; r_{21} R_{10}^n \; r_{21} R_{01}^n \ldots r_{2,2k-3} R_{2k-3,0}^n \\ \ldots r_{2,2k-3} R_{0,2k-3}^n \;\; 0 \ldots 0\big] \,|_{(x_{2k}, y_{2k})},$$

and

$$\mathcal{R}_2^{2k} = \big[r_{20} R_{00}^n \; r_{21} R_{10}^n \; r_{21} R_{01}^n \ldots r_{2,2k-3} R_{2k-3,0}^n \\ \ldots r_{2,2k-3} R_{0,2k-3}^n \; 0 \ldots 0\big] \,|_{(x_{2k-1}, y_{2k-1})},$$

where $n = 2k - 2$ and $\mathcal{R}_{ij}^n$ is given in (34). As earlier, the entries of the above rows corresponding to R_{ij}^n with $i + j = 2k - 2$ were eliminated, since the monomials ϕ_{ij} there belong to the space of interpolating polynomials: Π_{2k-2}°.

Before we use the Laplace theorem for the determinant of $\mathbf{V}''$, we eliminate, in the same way as in the previous case, also the entries of the above two rows corresponding to R_{ij}^n with $i + j = 2k - 3, 2k - 4, \ldots, 2k - k = k$. Thus finally they will be reduced to the following two rows, where $n = 2k - 2$ and a_i^2 are some numbers:

$$\mathcal{R}_2^{2k+1} := \left[a_0^2 R_{00}^n \; a_1^2 R_{10}^n \; a_1^2 R_{01}^n \ldots a_{k-1}^2 R_{k-10}^n \ldots a_{k-1}^2 R_{0k-1}^n \; 0 \ldots 0\right] |_{(x_{2k}, y_{2k})},$$

$$\mathcal{R}_2^{2k} := \left[a_0^2 R_{00}^n \;\; a_1^2 R_{10}^n \; a_1^2 R_{01}^n \ldots a_{k-1}^2 R_{k-10}^n \ldots a_{k-1}^2 R_{0k-1}^n \; 0 \ldots 0\right] |_{(x_{2k-1}, y_{2k-1})}.$$

Note that the coefficients a_i^2 are the same for these two rows. Also they are the same for the entries R_{ij}^n with $i + j = s$. Let us mention that the entries of these rows, preceding the last zeros, will remain unchanged till the end of this step of factorizations.

We get by using the Laplace theorem:

$$\det \mathbf{V}'' = \rho''' \det \mathbf{V}''' \det V_{2k-2,2k-2}^\circ, \tag{36}$$

where $\rho''' = \rho'''(r_1, \mu_1, \ldots, r_s, \mu_s)$ and the matrix $\mathbf{V}'''$ is obtained from $\mathbf{V}''$ by replacing its second block row by the above rows $\mathcal{R}_2^{2k+1}, \mathcal{R}_2^{2k}$ and by canceling the last block column ($2k - 1$ columns).

Now by combining (28), (35), and (36) we get

$$\det \mathbf{V}(1) = \rho' \rho'' \rho''' \det \mathbf{V}''' \det V_{2k-2,2k-2}^\circ \det V_{2k-1,2k-1}^\circ.$$

Continuing this way, after the last kth factorization, we get

$$\det \mathbf{V}(1) = \rho \prod_{i=k}^{2k-1} \det(V_i^\circ) \;\; \det(\mathbf{A}), \tag{37}$$

where $\rho = \rho(r_1, \mu_1, \dots, r_s, \mu_s)$ and **A** is the following matrix

$$\mathbf{A} = \begin{bmatrix} a_0^1 R_{00}^{2k-1} & a_1^1 R_{10}^{2k-1} & a_1^1 R_{01}^{2k-1} & \cdots & a_{k-1}^1 R_{k-1,0}^{2k-1} & \cdots & a_{k-1}^1 R_{0,k-1}^{2k-1} \\ a_0^2 R_{00}^{2k-2} & a_1^2 R_{10}^{2k-2} & a_1^2 R_{01}^{2k-2} & \cdots & a_{k-1}^2 R_{k-1,0}^{2k-2} & \cdots & a_{k-1}^2 R_{0,k-1}^{2k-2} \\ a_0^2 R_{00}^{2k-2} & a_1^2 R_{10}^{2k-2} & a_1^2 R_{01}^{2k-2} & \cdots & a_{k-1}^2 R_{k-1,0}^{2k-2} & \cdots & a_{k-1}^2 R_{0,k-1}^{2k-2} \\ & \cdots & & \cdots & & \cdots & \\ a_0^k R_{00}^k & a_1^k R_{10}^k & a_1^k R_{01}^k & \cdots & a_{k-1}^k R_{k-1,0}^k & \cdots & a_{k-1}^k R_{0,k-1}^k \\ & \cdots & & \cdots & & \cdots & \\ a_0^k R_{00}^k & a_1^k R_{10}^k & a_1^k R_{01}^k & \cdots & a_{k-1}^k R_{k-1,0}^k & \cdots & a_{k-1}^k R_{0,k-1}^k \end{bmatrix}.$$

Here the first row is evaluated at (x_{2k}, y_{2k}), second and third rows at (x_{2k}, y_{2k}) and (x_{2k-1}, y_{2k-1}), respectively, and the last k rows at $(x_{2k}, y_{2k}), \dots, (x_{k+1}, y_{k+1})$, respectively. Note also that the coefficients of the entries with R_{ij}^n depend only on n and $(i + j)$.

Step 3: Factorizations A. Note that so far all we used for the set of basic nodes was that $\mathcal{B}_{2k-1}$ does not contain collinear nodes. At this step we will use the condition that

$$\mathcal{B} \subset C_2^\circ,$$

where C_2° is given in (4).

Here we will factorize $\det(\mathbf{A})$ by using the Laplace theorem with respect to the last k rows. Beforehand we will eliminate some entries there. Denote the submatrix of **A** formed by its last k rows by B. Note that the submatrix in the last k columns of B is $a_{k-1}^k A_{k-1}$, where the matrix A_{k-1} is given in (22).

Our first aim is to eliminate by elementary row operations of **A** all the columns of B except the last k ones. We begin by proving that $a_{k-1}^k \neq 0$. Conversely suppose that $a_{k-1}^k = 0$. Then according to the statement (14), the last $2k-1$ columns of B vanish, or in other words, all the entries of B corresponding to R_{ij}^k with $i + j \geqslant k - 2$ vanish. Next, the statements (14) and (15), with $m + 2l = k$ or $k - 3$, imply that all the columns of B are either zero or linear combinations of its $k-2$ columns with entries corresponding to $R_{k-3,0}^k, \dots, R_{0,k-3}^k$, respectively. This means that the maximal number of linearly independent columns and therefore rows of B is $\leqslant k-2$. Therefore the rows of B and hence the rows of **A** are linearly dependent, which contradicts to (19).

Thus, we have $a_{k-1}^k \neq 0$. By using again the statements (14) and (15) now with $m+2l = k-1$ we get that all the columns of B are (either zero or) linear combinations of its last k columns. This enables us to carry out the elimination of those columns of B mentioned above as follows. Consider such a column with entries corresponding to $R_{\mu\nu}^k(x_l, y_l)$, with $\mu + \nu \leqslant k - 2$. If $k - \mu - \nu$ is even then in view of (14) the column vanishes. Otherwise, according to the above conclusion:

$$R_{\mu\nu}^j(x_l, y_l) = \sum_{i=0}^{k-1} c_i R_{k-1-i,i}^j(x_l, y_l),$$

where $l = k + 1, \dots, 2k + 1, \quad j = k, \dots, 2k - 1$, and c_i depends only on μ, ν, k and C_2° not on j or l.

Now let us perform the following column operation of the matrix **A** :

$$\mathcal{C}_{\mu\nu} \to \mathcal{C}_{\mu\nu} - \frac{a^k_{\mu+\nu}}{a^k_{k-1}} \sum_{i=0}^{k-1} c_i \mathcal{C}_{k-1-i,i},$$

where $\mathcal{C}_{mn}$ is the column of **A** with entries corresponding to R^j_{mn}, and the coefficients c_i are from the above relation. It is easily seen that the column of **A** being considered becomes then

$$\mathcal{C}_{\mu\nu} = \left[\tilde{a}^1_s R^{2k-1}_{\mu\nu}, \tilde{a}^2_s R^{2k-2}_{\mu\nu},\ \tilde{a}^2_s R^{2k-2}_{\mu\nu}, \ldots, \tilde{a}^{k-1}_s R^{k+1}_{\mu\nu}, \ldots, \tilde{a}^{k-1}_s R^{k+1}_{\mu\nu}, 0 \ldots, 0\right]^T,$$

where $s = \mu + \nu$ and

$$\tilde{a}^j_s = a^j_s - a^j_{k-1} \frac{a^k_s}{a^k_{k-1}}.$$

From here we conclude, what is important, that the new coefficients are the same for all columns $\mathcal{C}_{\mu\nu}$ with $s = \mu + \nu$ and also they are the same for entries corresponding to $R^l_{\mu\nu}$ with $s = \mu + \nu$. Let us mention also that $\tilde{a}^j_s$ depend only on $r_1, \mu_1 \ldots, r_s \mu_s$, not on the conic section C°_2. In other words the property of the coefficients of the matrix **A** mentioned just before Step 3 is preserved.

Now the Laplace theorem gives

$$\det \mathbf{A} = (a^{k-1}_k)^k \det A_{k-1} \det \tilde{\mathbf{A}},$$

where

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{a}^1_0 R^{2k-1}_{00} & \tilde{a}^1_1 R^{2k-1}_{10} & \tilde{a}^1_1 R^{2k-1}_{01} & \cdots & \tilde{a}^1_{k-2} R^{2k-1}_{k-2,0} & \cdots & \tilde{a}^1_{k-2} R^{2k-1}_{0,k-2} \\ \tilde{a}^2_0 R^{2k-2}_{00} & \tilde{a}^2_1 R^{2k-2}_{10} & \tilde{a}^2_1 R^{2k-2}_{01} & \cdots & \tilde{a}^2_{k-2} R^{2k-2}_{k-2,0} & \cdots & \tilde{a}^2_{k-2} R^{2k-2}_{0,k-2} \\ \tilde{a}^2_0 R^{2k-2}_{00} & \tilde{a}^2_1 R^{2k-2}_{10} & \tilde{a}^2_1 R^{2k-2}_{01} & \cdots & \tilde{a}^2_{k-2} R^{2k-2}_{k-2,0} & \cdots & \tilde{a}^2_{k-2} R^{2k-2}_{0,k-2} \\ & \cdots & & \cdots & & \cdots & \\ \tilde{a}^{k+1}_0 R^{k+1}_{00} & \tilde{a}^{k+1}_1 R^{k+1}_{10} & \tilde{a}^{k+1}_1 R^{k+1}_{01} & \cdots & \tilde{a}^{k+1}_{k-2} R^{k+1}_{k-2,0} & \cdots & \tilde{a}^{k+1}_{k-2} R^{k+1}_{0,k-2} \\ & \cdots & & \cdots & & \cdots & \\ \tilde{a}^{k+1}_0 R^{k+1}_{00} & \tilde{a}^{k+1}_1 R^{k+1}_{10} & \tilde{a}^{k+1}_1 R^{k+1}_{01} & \cdots & \tilde{a}^{k+1}_{k-2} R^{k+1}_{k-2,0} & \cdots & \tilde{a}^{k+1}_{k-2} R^{k+1}_{0,k-2} \end{bmatrix}.$$

Continuing this way we get the factorization

$$\det \mathbf{A} = \bar{\rho} \prod_{i=0}^{k-1} \det(A_i),$$

where $\bar{\rho} = \bar{\rho}(r_1, \mu_1, \ldots, r_s, \mu_s)$. This combined with (37) yields the desired formula (23), where $\rho_\delta = \rho\bar{\rho} \neq 0$ according to the statement (19). This completes the proof for the case $\delta = 1$.

Part 2: The case $\delta = 0$. This case can be reduced to the previous case $\delta = 1$. We transform $\mathbf{V}(0)$, in the same way as in Part 1, to the form (27). Now we get

$$\det \mathbf{V}(0) = \rho' \det \mathbf{V}',$$

where the matrix $\mathbf{V}'$ in this case is

$$\mathbf{V}' = \begin{bmatrix} r_{11}V^\circ_{2k,0} & & \cdots & & r_{1,2k}V^\circ_{2k,2k-1} & V^\circ_{2k,2k} \\ r_{21}V^\circ_{2k,0} & \cdots & & r_{2,2k-1}V^\circ_{2k,2k-2} & V^\circ_{2k,2k-1} & 0 \\ \cdots & & & \cdots & & \cdots \\ r_{k+1,1}V^\circ_{2k,0} & \cdots\ r_{k+1,k-1}V^\circ_{2k,k-1} & V^\circ_{2k,k} & 0 & 0 & 0 \end{bmatrix}.$$

Note that ρ' and r_{ij} depend only on $r_1, \mu_1, \ldots, r_s, \mu_s$.

Then notice that the nonzero elements in the last block column above form a submatrix of dimension $(2k+1) \times (2k+1)$. Thus without additional undertaking we get from the Laplace theorem

$$\det \mathbf{V}' = \det \mathbf{V}'' \det V^\circ_{2k,2k},$$

where $\mathbf{V}''$ is exactly of form (27). Therefore it remains to apply the result of Part 1. This completes the proof. $\square$

Proof of Theorem 6. Suppose that $\mathcal{B} = \mathcal{B}_{2k}$ does not contain $\delta+1$ pairs of opposite nodes. Then there are no opposite nodes in the case $\delta = 0$ and without loss of generality we can assume that there are no opposite nodes inside $\mathcal{B}_{2k-1}$ in the case $\delta = 1$ as well. Thus in both cases the determinants V_i° in the right side of the formula (23) do not vanish. This means that the Bojanov–Xu interpolation for $\delta = 0, 1$ is poised if and only if the determinants A_i there do not vanish.

Now what remains is to show that $\det A_m \neq 0$ if and only if the $\{m, n\}$-degree interpolation ($n = 2k - m - 1$) is poised, for each $m = 0, \ldots, k-1$. Thus, suppose that $\det A_m = 0$ for some fixed m (see (22)). Then the columns of the matrix A_m are linearly dependent:

$$\sum_{i=0}^{m} \alpha_i \mathcal{C}_i = 0,$$

where $\mathcal{C}_i$ is the ith column and not all the coefficients are zero. This implies that

$$R^\circ_{n,g}(x_j, y_j) = 0, \quad j = n+1, \ldots, 2k,$$

where

$$g = \sum_{i=0}^{m} \alpha_i g_i = \sum_{i=0}^{m} \alpha_i x^{m-i} y^i \not\equiv 0.$$

On the other hand, by the notion of the remainder and (9),

$$R^\circ_{n,g}(x_j, y_j) = 0, \quad j = 0, \ldots, n.$$

Thus

$$g(x_j, y_j) - P^\circ_{n,g}(x_j, y_j) = 0 \quad \text{for } j = 0, \ldots, 2k. \tag{38}$$

Now, notice that the polynomial in the left side of this equality is not identically zero, belongs to $\Pi^\circ_m \oplus \Pi^\circ_n$ and vanishes at all the nodes. This means that the $\{m, n\}$-degree interpolation is not poised.

Next assume that the $\{m, n\}$-degree interpolation is not poised. Then

$$g(x_j, y_j) - f(x_j, y_j) = 0 \text{ for } j = 0, \ldots, 2k,$$

where $g \in \Pi_m^\circ, \quad f \in \Pi_n^\circ$, and $g \not\equiv 0$. This implies that

$$f = P_{n,g}^\circ$$

and the relation (38) takes place. The latter, as we knew, is equivalent to $\det A_m = 0$.

Proof of Theorem 11. Assume that $\mathcal{B}$ satisfies the opposite node property. Suppose without loss of generality that $\mathcal{B}_{2k-1}$ does not contain opposite nodes. Let $\mathcal{B}_k^* = \{N_i\}_{i=0}^k \subset \mathcal{B}$ be the set from Definition 10. We will prove that each $\{m, n\}$-degree interpolation is poised, where $m = 0, \ldots, k-1, \; n = 2k - m - 1$. Fix any such m. Assume that

$$(g - f)(x_i, y_i) = 0, \quad \text{for} \quad i = 0, \ldots, 2k,$$

where $g \in \Pi_m^\circ$ and $f \in \Pi_n^\circ$. Then it suffices to show

$$g, f \equiv 0.$$

Thus we have that

$$g(x_i, y_i) = f(x_i, y_i) \quad \text{for} \quad i = 0, \ldots, 2k. \tag{39}$$

Let us first consider the case when

$$g(x_i, y_i) = f(x_i, y_i) \neq 0, \quad i = 0, \ldots, 2k.$$

We are going to show that on each open arc: $(\gamma_s) = \widehat{N_{s-1}, N_s}, \; s = 1, \ldots, k$, between two neighbor basic nodes of $\mathcal{B}_k^*$, the total number of zeros of g and f is at least 2. Let us fix such an s. At the endpoints of the arc (γ_s) : N_{s-1} and N_s, the polynomials $g, \; f$ assume the same values, say α and β, respectively. If $\alpha\beta < 0$ then each of the two polynomials will have zero inside the open arc. Thus suppose $\alpha\beta > 0$. Let $N_\sigma = (x_\sigma, y_\sigma)$, $\sigma = k + s$, be the node from $\mathcal{B}$ such that $-N_\sigma \in (\gamma_s)$ (see Definition 10 (c)).

Now by denoting $g(x_\sigma, y_\sigma) = f(x_\sigma, y_\sigma) := \gamma$ we get that

$$g(-x_\sigma, -y_\sigma) = (-1)^m \gamma \quad \text{and} \quad f(-x_\sigma, -y_\sigma) = (-1)^n \gamma.$$

Therefore the values of g and f at the three successive points $N_{s-1}, \; -N_\sigma, \; N_s$ of the arc γ_s are

$$\alpha, \quad (-1)^m \gamma, \quad \beta, \quad \text{and} \quad \alpha, \quad (-1)^n \gamma, \quad \beta,$$

respectively. The mean terms here have different signs, since the numbers $m, \; n$, have different parity, while the first and third terms: α, β, have the same sign. Thus one of the polynomials changes its sign on the considered arc at least twice and therefore has at least two zeros.

Summarizing, we have that the total number of zeros of homogeneous polynomials g and f is at least $m + n + 1 = 2k$. Therefore either g has more than m zeros or f has more than

n zeros. Notice that, by virtue of the condition (b) of Definition 10, there are no opposite zeros. Therefore $g \equiv 0$ or $f \equiv 0$, respectively. Now the condition (39) readily implies that the other one is also identical to zero.

Next let us return to the condition (39) and consider the case when

$$f(x_j, y_j) = g(x_j, y_j) = 0$$

for some fixed $j = 0, \ldots, 2k$. Then

$$g(x, y) = (ax + by)g_1(x, y) \quad \text{and} \quad f(x, y) = (ax + by)f_1(x, y),$$

where the line with the equation $ax + by = 0$ passes through (x_j, y_j).

Thus the given problem is reduced to

$$(g_1 - f_1)(x_i, y_i) = 0 \quad \text{for} \quad i = 0, \ldots, 2k,\ i \neq j, \tag{40}$$

where $g_1 \in \Pi^\circ_{m-1}$ and $f_1 \in \Pi^\circ_{n-1}$ and we are to show that

$$g_1, f_1 \equiv 0.$$

In other words, k was replaced by $k - 1$ (one equality in (40) is extra and can be ignored).

The only point remains is to verify that the opposite node property holds with the latter problem. First consider the case when no point of $\mathcal{B}$ is opposite to (x_j, y_j). Then the subset in Definition 10 can be chosen as $\mathcal{B}^*_{k-1} := \mathcal{B}^*_k \setminus \{(x_s, y_s)\}$, where

$$s = \begin{cases} j & \text{if } (x_j, y_j) \in \mathcal{B}^*_k, \\ j - k & \text{if } -(x_j, y_j) \in \widehat{N_{j-k+1}, N_{j-k}}. \end{cases}$$

Also the equality with $i = j + k$ or $j - k$ can be ignored in (40), respectively.

Now consider the case when there is a point of $\mathcal{B}$ opposite to (x_j, y_j). Note that this is possible only in the case $\delta = 1$. Then it is easily seen that (x_j, y_j) and its opposite point necessarily coincide with the nodes $N_0, N_k \in \mathcal{B}^*_k$, where

$$N_0 = -N_k.$$

Finally what remains is to note that the subset of Definition 10 in this case can be taken as $\mathcal{B}^*_{k-1} = \mathcal{B} \setminus \mathcal{B}^*_k$. Also the equality with $i = k$ or $i = 0$ can be ignored in (40), if $j = 0$ or $j = k$, respectively. This completes the proof. $\square$

Proof of Corollary 12. Throughout the proof the arc with angle π : $\widehat{N, -N}$ means the one for which N goes to $-N$ counterclockwise. Suppose that $\mathcal{B} = \{N_i\}_{i=0}^{2k}$, where $N_0, \ldots, N_{2k}$ are lying successively counterclockwise on the ellipse E_2, given in (5).

First consider the case when $\mathcal{B}$ contains a pair of opposite nodes: N and $-N$. Note that this is possible only in the case $\delta = 1$. Then one of the arcs $\widehat{N, -N}$, $\widehat{-N, N}$ contains $\leqslant (k+1)$ and another $\geqslant (k+2)$ nodes (the nodes N and $-N$ are counted in both cases). Therefore if one of the conditions of Corollary 12: (i) or (ii) is satisfied then these numbers become $(k+1)$ and $(k+2)$. Moreover, the $(k+1)$ nodes or the $(k+2)$ nodes become equidistant, with arcs between neighbors equal π/k or $\pi/(k+1)$, respectively. Now let us choose the $(k+1)$ nodes to form the set $\mathcal{B}^*_k = \{N_i\}_{i=0}^{k}$ from Definition 10. In view of Theorem 11, it suffices to show that $\mathcal{B}$ satisfies the opposite node property. For this one needs only to verify the condition (c) of Definition 10.

If an arc $\gamma_j = \widehat{N_{j-1}, N_j}$, $\quad j = 1, \ldots, k$, does not contain opposite of a node, then $-\gamma_j = \widehat{-N_{j-1}, -N_j}$ does not contain any of the above $(k+2)$ nodes. Let γ_* be the arc with neighboring nodes which contains the arc $\widehat{-N_{j-1}, -N_j}$.

Consider the case when the above-mentioned condition (i) holds. Then, as was stated above, $\angle\gamma_j = \pi/k$, where $\angle$ means the angle. Now we get $\angle\gamma_* > \angle\{-\gamma_j\} = \pi/k$, which contradicts the condition (i).

Next assume that the condition (ii) holds. Then, correspondingly, $\angle\gamma_* = \pi/(k+1)$ and we get $\angle\gamma_j = \angle\{-\gamma_j\} < \angle\gamma_* = \pi/(k+1)$. This contradicts the condition (ii).

Now consider the case when there are no opposite nodes. Then it is enough to prove that for each arc $\gamma_i = \widehat{N_{i-1}, N_i}$ with the neighboring nodes from $\mathcal{B}$, there is a node $N \in \mathcal{B}$ such that $-N \in (\gamma_i)$, $\quad i = 1, \ldots, 2k$. Indeed, then one of the arcs $\widehat{N_0, N_k}$ and $\widehat{N_k, N_{2k}}$, which is $< \pi$ can be chosen as the arc Γ of Definition 10.

Conversely assume that this is not satisfied for some fixed i. This means that the arc $\widehat{-N_{i-1}, -N_i}$ does not contain nodes from $\mathcal{B}$. In other words the basic nodes belong to the arcs $\widehat{N_{i-1}, -N_{i-1}}$ and $\widehat{-N_i, N_i}$. Both these arcs contain the arc $\widehat{N_{i-1}, N_i}$. This means that the total number of nodes on these arcs is $2k+3$. Hence one of these arcs contains $\leqslant (k+1)$ nodes and another $\geqslant (k+2)$ nodes. In other words there are $\leqslant k$ and $\geqslant (k+1)$ arcs with neigbouring nodes on them, respectively. Therefore we conclude that there are two arcs with neighboring nodes having angles one $\leqslant \pi/(k+1)$ and another $\geqslant \pi/k$. But we can sharpen these estimates by making the arcs $\widehat{N_{i-1}, -N_{i-1}}$ and $\widehat{-N_i, N_i}$ less than π, by shifting $-N_{i-1}$ and $-N_i$ a bit clockwise and counterclockwise, respectively. This is allowed since $-N_{i-1}$ and $-N_i$ do not coincide with any node in this case. This completes the proof. $\square$

Proof of Lemma 16. Here we will use the following statement

$$p \in \Pi_m^\circ \oplus \Pi_n^\circ \text{ vanishes identically on } C_2^\circ \quad \Rightarrow \quad p \equiv 0, \tag{41}$$

where $m+n = 2k-1$. Indeed, suppose that $p = p_m + p_n$, where $p_m \in \Pi_m^\circ$, and $p_n \in \Pi_n^\circ$, vanishes identically on C_2° given by (4). Then by the Bézout theorem we get

$$p(x, y) = (\alpha x^2 + \beta xy + \gamma y^2 - 1) r(x, y).$$

Suppose m is odd, then n is even. By comparing the terms with odd and even total degrees in both sides of the above equality we get

$$\begin{aligned} p_m(x, y) &= (\alpha x^2 + \beta xy + \gamma y^2 - 1) r_1(x, y), \\ p_n(x, y) &= (\alpha x^2 + \beta xy + \gamma y^2 - 1) r_2(x, y), \end{aligned} \tag{42}$$

where r_1 and r_2 are composed by the terms of r with odd and even total degrees, respectively. In particular $r = r_1 + r_2$. Finally notice that, unless $r_1 \equiv 0$ and $r_2 \equiv 0$, the difference of maximum and minimum total degrees of monomials of each of the homogeneous polynomials p_m and p_n, according to (42), is at least 2, which is a contradiction.

Now let us turn to Lemma. Suppose the $\{m, n\}$-degree interpolation, where $n = 2k-m-1$, is not poised. Then there is a nonzero polynomial

$$p \in \Pi_m^\circ \oplus \Pi_n^\circ, \tag{43}$$

such that

$$p(x_i, y_i) = p(x_i, -y_i) = 0, \quad i = 1, \dots, k \quad \text{and} \quad p(a, 0) = 0. \tag{44}$$

Suppose

$$p(x, y) = \sum_{i=0}^{m} \alpha_i x^{m-i} y^i + \sum_{i=0}^{n} \beta_i x^{n-i} y^i.$$

Consider the polynomials

$$p_1(x, y) := \frac{1}{2}\left[p(x, y) + p(x, -y)\right], \quad \tilde{p}_2(x, y) := \frac{1}{2}\left[p(x, y) - p(x, -y)\right].$$

Notice that both of them satisfy the above conditions (43)–(44). Next we get

$$p_1(x, y) = \sum_{i=0}^{[m/2]} \alpha_i x^{m-2i} y^{2i} + \sum_{i=0}^{[n/2]} \beta_i x^{n-2i} y^{2i}$$

and

$$\tilde{p}_2(x, y) = y \sum_{i=0}^{[(m-1)/2]} \alpha_i x^{m-2i-1} y^{2i} + y \sum_{i=0}^{[(n-1)/2]} \beta_i x^{n-2i-1} y^{2i} =: y p_2(x, y).$$

It is easily seen that

$$p_2 \in \Pi_{m-1}^\circ \oplus \Pi_{n-1}^\circ,$$

$$p_2(x_i, y_i) = p_2(x_i, -y_i) = 0, \quad i = 1, \dots, k.$$

Notice that at least one of the polynomials p_1, p_2 does not vanish identically, since $p = p_1 + y p_2$.

Now consider the polynomials

$$q_1(x) = \sum_{i=0}^{[m/2]} \alpha_i x^{m-2i} \left[\pm b^2 \left(1 - \frac{x^2}{a^2}\right)\right]^i + \sum_{i=0}^{[n/2]} \beta_i x^{n-2i} \left[\pm b^2 \left(1 - \frac{x^2}{a^2}\right)\right]^i$$

and

$$q_2(x) = \sum_{i=0}^{[(m-1)/2]} \alpha_i x^{m-2i-1} \left[\pm b^2 \left(1 - \frac{x^2}{a^2}\right)\right]^i + \sum_{i=0}^{[(n-1)/2]} \beta_i x^{n-2i-1} \left[\pm b^2 \left(1 - \frac{x^2}{a^2}\right)\right]^i.$$

The case of the ellipse E_2 and hyperbola H_2, given in (5)–(6), correspond to $(+)$ and $(-)$, respectively. We have that

$$q_1(x) = p_1(x, b\phi(x)) \quad \text{and} \quad q_2(x) = p_2(x, b\phi(x)),$$

where

$$\phi(x) := b\sqrt{\pm\left(1 - \frac{x^2}{a^2}\right)}. \tag{45}$$

Notice that q_1 and q_2 do not vanish identically at the same time. Indeed, otherwise p_1 and p_2 vanish identically on E_2 or H_2 and therefore, according to (41), $p_1, p_2 \equiv 0$, which, as was mentioned above, is not possible.

Then we have that $q_\varepsilon,\ \varepsilon = 0, 1$, has the form mentioned in Problem 13(ε) and satisfies the the homogeneous condition (24), that is, the condition

$$q_\varepsilon(x_i) = 0 \quad \text{for} \quad i = \varepsilon, \ldots, k. \tag{46}$$

Therefore we conclude that one of the interpolations in Problem 13(ε) corresponding to either $\varepsilon = 0$ or 1 is not poised.

Next assume conversely, that one of these two interpolations is not poised. Then there is a nonzero polynomial $q_\varepsilon,\ \varepsilon = 0$ or 1 of above-mentioned form

$$q_\varepsilon(x) = \sum_{i=0}^{m-\varepsilon} \alpha_i x^{m-i-\varepsilon} + \sum_{i=0}^{[(n-\varepsilon)/2]} \beta_i x^{n-2i-\varepsilon},$$

that satisfies the condition (46).

Consider the polynomial

$$p(x, y) := \sum_{i=0}^{m-\varepsilon} \alpha_i x^{m-i-\varepsilon}\left(\frac{x^2}{a^2} \pm \frac{y^2}{b^2}\right)^{\gamma_i} + \sum_{i=0}^{[(n-\varepsilon)/2]} \beta_i x^{n-2i-\varepsilon}\left(\frac{x^2}{a^2} \pm \frac{y^2}{b^2}\right)^{i},$$

where

$$\gamma_i = \begin{cases} (n - m + i)/2 & \text{if i is odd,} \\ i/2 & \text{if i is even.} \end{cases}$$

Let us mention that

$$p \in \Pi^\circ_{m-\varepsilon} \oplus \Pi^\circ_{n-\varepsilon}.$$

We claim that $p \neq 0$. Indeed, for this it is enough to notice that

$$q_\varepsilon(x) = p(x, \phi(x)),$$

where $\phi(x)$ is given in (45). Finally set

$$\tilde{p}(x, y) = y^\varepsilon p(x, y).$$

Then $\tilde{p} \neq 0$ and $\tilde{p} \in \Pi^\circ_m \oplus \Pi^\circ_n$. The only point remains is to note that, in view of the condition (46), the condition (44) is satisfied too. Thus the $\{m, n\}$-degree interpolation is not poised. This completes the proof. $\square$

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